Centre Number

Student Number

SCEGGS Darlinghurst

2003
HIGHER SCHOOL CERTIFICATE
TRIAL EXAMINATION

Mathematics Extension 1

This is a TRIAL PAPER only and does not necessarily reflect the content or format of the Higher School Certificate Examination for this subject.

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks - 84
- Attempt Questions 1–7
- All questions are of equal value

Total marks – 84
Attempt Questions 1–7
All questions are of equal value

Answer each question on a NEW page.

Marks

Question 1 (12 marks)

(a) Find the co-ordinates of the point P which divides the interval joining A(–3, 2) and B(5, 6) externally in the ratio 1:3. **2**

(b) Differentiate $x\tan^{-1}(x^2)$. **2**

(c) Find n if $\binom{n}{2}=91$. **2**

(d) Evaluate $\int_0^1 \frac{dx}{\sqrt{4-x^2}}$. **2**

(e) Given $x+3$ is a factor of

$$P(x)=x^3-Ax^2+2x-1 .$$

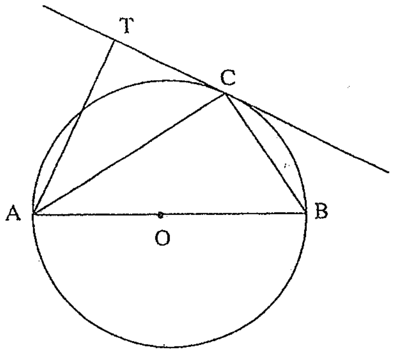
Find the value of A. **2**

(f) Explain why $\lim_{x\to 0}\frac{\sin 2x}{3x}=\frac{2}{3}$ **2**

Marks

Question 2 (12 marks) Start a NEW page.

(a) 3



NOT TO SCALE

AOB is the diameter of a circle centre O and C is the point of contact of the tangent TC such that AC bisects $\angle TAB$. Prove that AT is perpendicular to TC.

(b) How many arrangements of the letters of the word DEFINITION are there if the letters N are not together? 2

(c) Consider the function

$$y = \frac{1}{2}\cos^{-1}(2x - 1)$$

(i) Find its domain. 1

(ii) Sketch the function. 1

(iii) Evaluate y if $x = \frac{1}{4}$. 2

(d) When a biased coin is tossed it shows heads in 2 out of every 3 tosses. The coin is tossed 15 times. Find:

(i) the probability of 12 heads. 1

(ii) the probability of at least 2 heads. 2

You may leave your answers in index form.

Marks

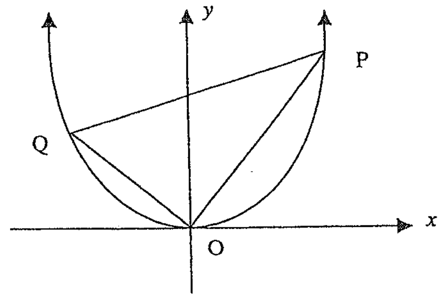
Question 3 (12 marks) Start a NEW page.

(a) Find the co-efficient of x in the expansion of $\left(x^2 - \frac{3}{x}\right)^8$. 3

(b) Evaluate $\int_0^3 \frac{x}{\sqrt{1+x}}\,dx$ using the substitution $x = u^2 - 1$. 3

(c)

NOT TO SCALE



$P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are points on the parabola $x^2 = 4ay$. It is given that $\angle POQ = 90°$. O is the origin.

(i) Prove that $pq = -4$ 2

(ii) Find the co-ordinates of M, the midpoint of PQ. 1

(iii) Prove that the Cartesian equation of the locus of M is 3

$$2ay = x^2 + 8a^2.$$

Marks

Question 4 (12 marks) Start a NEW page.

(a) If the equation $3x^3 - 4x^2 + 2x + 1 = 0$ has roots α, β and γ find:

(i) $2\alpha + 2\beta + 2\gamma$. **1**

(ii) the equation whose roots are $2\alpha, 2\beta$ and 2γ . **2**

(b) A metal sphere is heated such that the surface area is increasing at 4π mm^2 per minute.

(i) Find the rate of increase of the radius when the radius is 20mm. **3**

(ii) Find the rate of increase of the volume at this time. **2**

(c) Use Mathematical Induction to prove that **4**

$$2\times 5 + 4\times 8 + \ldots + 2n(3n+2) = n(n+1)(2n+3)$$

for all positive integers n.

Marks

Question 5 (12 marks) Start a NEW page.

(a) (i) Prove that $\dfrac{1}{1+\cos x} = \dfrac{1}{2}\sec^2\dfrac{x}{2}$ **2**

(ii) Hence evaluate $\displaystyle\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{dx}{1+\cos x}$ **2**

(b) A committee of 5 is to be formed from a group of 9 men and 7 women.

(i) How many different committees are possible? **1**

(ii) How many would include only 3 women? **1**

(iii) How many are possible if Mr and Mrs Brown cannot both serve? **2**

(c) Use the substitution $u = \ln x$ to evaluate **4**

$$\int_1^e \frac{dx}{x(1+2\ln x)^2}$$

Marks

Question 6 (12 marks) Start a NEW page.

(a) (i) Sketch $y = f(x)$ if $f(x) = e^{x+2}$ **1**

(ii) Find the inverse function $f^{-1}(x)$. **2**

(iii) Sketch $y = f^{-1}(x)$. You may choose to do this on your sketch in part (i). **1**

(b) A particle moves in a straight line so that when it is x metres from the origin O its velocity v m/s is given by

$$v^2 = 32 + 8x - 4x^2$$

(i) Prove that the particle is moving in Simple Harmonic Motion. **2**

(ii) Find the centre of the motion. **1**

(iii) Find the period and amplitude of the motion. **2**

(iv) Given that the particle is initially at $x = 4$, which of the 2 equations could describe its motion: **1**

$$x = 1 + 3\sin 2t$$

or

$$x = 1 + 3\cos 2t$$

(v) When does the particle pass through the origin for the first time? **2**

Marks

Question 7 (12 marks) Start a NEW page.

(a) Evaluate exactly $\sin\left(2\cos^{-1}\frac{2}{3}\right)$ **2**

(b) (i) Express $\sqrt{3}\sin x - \cos x$ in the form $A\sin(x - \alpha)$ if $A > 0$ and $0 \le \alpha \le \frac{\pi}{2}$ **2**

(ii) Hence sketch the curve **3**

$$y = \sqrt{3}\sin x - \cos x \quad \text{for } 0 \le x \le 2\pi$$

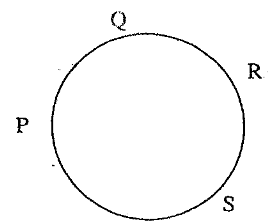
showing all important features.

(iii) Use your sketch to determine the value(s) of k for which **1**

$$\sqrt{3}\sin x - \cos x = k$$

has 3 distinct solutions for $0 \le x \le 2\pi$

(c) **4**



NOT TO SCALE

P, Q, R and S are points on the circumference of a circle.

Prove that $\dfrac{PR}{QS} = \dfrac{\sin \angle PQR}{\sin \angle QPS}$

END OF PAPER

Extension 1 Trial 2003 SCEGGS.

1) a) $A(-3,2)$ $B(5,6)$ ratio $-1:3$

$x = \frac{-3\times3+5\times-1}{-1+3}$

$= \frac{-14}{2}$ ✓

$y = \frac{2\times3+6\times-1}{2}$

$= 0$ ✓

$\therefore P$ is $(-7,0)$

b) $\frac{d}{dx}\left[x\tan^{-1}(x^2)\right]$

$= \tan^{-1}(x^2) + \frac{x\times2x}{1+x^2}$ ✓

$= \tan^{-1}(x^2) + \frac{2x^2}{1+x^2}$ ✓✓

c) $\frac{n!}{(n-2)!\,2!} = 91$

$\therefore n(n-1) = 182$ ✓

$n^2 - n - 182 = 0$

$(n-14)(n+13) = 0$

$n = 14$ ✓

d) $\int_0^1 \frac{dx}{\sqrt{4-x^2}} = \left[\sin^{-1}\frac{x}{2}\right]_0^1$ ✓

$= \sin^{-1}\frac{1}{2} - \sin^{-1}0$

$= \frac{\pi}{6}$ ✓

e) $P(-3) = -27 - 9A - 6 - 1 = 0$ ✓

$\therefore 9A = -34$

$A = -3\frac{7}{9}$ ✓

f) $\lim_{x\to0}\frac{\sin2x}{3x} = \lim_{x\to0}\frac{\sin2x}{2x}\times\frac{2}{3}$ ✓

since $\lim_{x\to0}\frac{\sin2x}{2x} = 1$, ✓

$\lim_{x\to0}\frac{\sin2x}{3x} = 1\times\frac{2}{3} = \frac{2}{3}$

(2) a)

$\angle TCA = \angle CBA$ (angle between tangent and chord equals angle in alternate segment)

$\angle TAC = \angle CAB$ (AC bisects $\angle TAB$, given)

$\angle ACB = 90°$ (angle in semi circle is $90°$)

$\Delta TAC\ |||\ \Delta ACB$ (equiangular)

$\therefore \angle ATC = \angle ACB$ (corresp $\angle$'s in sim Δ's) ✓

$\therefore AT \perp TC$

b) no. with H's together $= \frac{9!}{3!}$ ✓

total no. $= \frac{10!}{3!\,2!}$ ✓

$\therefore$ no. with H's apart $= \frac{10!}{2!\,3!} - \frac{9!}{3!}$

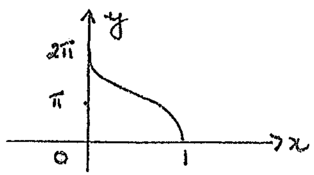
$= 241920$

c) (i) $-1 \le 2x-1 \le 1$

$0 \le 2x \le 2$

Domain: $0 \le x \le 1$ ✓

(ii) ✓



(iii) $y = \frac{1}{2}\cos^{-1}(\frac{1}{2}-1)$

$= \frac{1}{2}\cos^{-1}(-\frac{1}{2})$ ✓

$= \frac{1}{2}(\pi - \frac{\pi}{3})$

$= \frac{\pi}{3}$ ✓

d) $P(H) = p = \frac{2}{3}$

$P(T) = q = \frac{1}{3}$

(i) $P(12H) = \binom{15}{12}\left(\frac{2}{3}\right)^{12}\left(\frac{1}{3}\right)^3$ ✓

$P(\text{at least } 2H) = 1 - P(0H) - P(1H)$

$= 1 - \binom{15}{0}\left(\frac{1}{3}\right)^{15} - \binom{15}{1}\left(\frac{1}{3}\right)^{14}\left(\frac{2}{3}\right)^1$ ✓✓

(3) a) $T_{k+1} = \binom{8}{k}(x^2)^{8-k}(-3x^{-1})^k$

$= \binom{8}{k}x^{16-2k}(-3)^k x^{-k}$

$= \binom{8}{k}(-3)^k x^{16-3k}$ ✓

$\therefore 16 - 3k = 1$

$3k = 15$

$k = 5$ ✓

coefficient is $\binom{8}{5}(-3)^5$

$= -13608$ ✓

b) $x = u^2 - 1$

$dx = 2u\,du$

if $x = 3$, $u = 2$ ✓

$x = 0$, $u = 1$

$\int_0^3 \frac{x\,dx}{\sqrt{x+1}} = \int_1^2 \frac{u^2-1}{u}\times 2u\,du$

$= 2\int_1^2 u^2 - 1\,du$ ✓

$= 2\left[\frac{u^3}{3} - u\right]_1^2$

$= 2\left(\frac{8}{3} - 2 - \frac{1}{3} + 1\right)$

$\int_0^3 \frac{x\,dx}{\sqrt{x+1}} = \frac{8}{3}$ ✓

c) (i) gradient $PO = \frac{ap^2}{2ap} = \frac{p}{2}$

gradient $QO = \frac{q}{2}$ ✓

since $PO \perp OQ$, $\frac{p}{2}\times\frac{q}{2} = -1$ ✓

$pq = -4$

(ii) M is $\left(ap+aq, \frac{ap^2+aq^2}{2}\right)$ ✓

(iii) $x^2 = a^2p^2 + a^2q^2 + 2a^2pq$

$= a^2p^2 + a^2q^2 - 8a^2$

$\therefore x^2 + 8a^2 = a^2p^2 + a^2q^2$

$2ay = 2a.\frac{ap^2+aq^2}{2}$

$= a^2p^2 + a^2q^2$

$\therefore 2ay = x^2 + 8a^2$ is locus of M

(4) a) (i) $\alpha+\beta+\gamma = \frac{4}{3}$

$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{2}{3}$, $\alpha\beta\gamma = -\frac{1}{3}$

$\therefore 2(\alpha+\beta+\gamma) = \frac{8}{3}$ ✓

(ii) equation is:-

$x^3 - (2\alpha+2\beta+2\gamma)x^2 + (4\alpha\beta+4\alpha\gamma+4\beta\gamma)x - \ldots$ ✓

$x^3 - \frac{8}{3}x^2 + \frac{8x}{3} + \frac{8}{3} = 0$

or $3x^3 - 8x^2 + 8x + 8 = 0$ ✓

c) Consider $n=1$,

$LHS = 2\times5 = 10$

$RHS = 1(2)(5) = 10$

$\therefore$ true for $n=1$

Assume true for $n = k$

$\therefore 2\times5 + 4\times8 + \cdots + 2k(3k+2) = k(k+1)(2k+3)$

Consider $n = k+1$

$LHS = 2\times5 + \cdots + 2k(3k+2) + 2(k+1)(3k+5)$

$= k(k+1)(2k+3) + 2(k+1)(3k+5)$

$= (k+1)[2k^2 + 3k + 6k + 10]$

$= (k+1)(2k^2 + 9k + 10)$

$= (k+1)(k+2)(2k+5)$

$= RHS$ if $n = k+1$.

$\therefore$ if true for $n = k$ it is also true for $n = k+1$. It is true for $n = 1$ and thus for $n = 2, 3 \cdots$ i.e. true for all n positive integers.

5) a) (i) $1 + \cos 2x = 2\cos^2 x$

$\therefore 1 + \cos x = 2\cos^2 \frac{x}{2}$

$\therefore \frac{1}{1+\cos x} = \frac{1}{2}\sec^2 \frac{x}{2}$

(ii) $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{dx}{1+\cos x} = \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sec^2 \frac{x}{2}\, dx$

$= \left[\tan \frac{x}{2}\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$ ✓

$= \tan\frac{\pi}{4} - \tan\frac{\pi}{6}$

$= 1 - \frac{1}{\sqrt{3}}$ ✓

b) (i) no. possible $= \binom{16}{5} = 4368$ ✓

(ii) no. (3 women) $= \binom{7}{3}\times\binom{9}{2} = 1260$ ✓

(iii) no. with both $= \binom{14}{3}$

$= 364$ ✓

$\therefore$ no required $= 4368 - 364$

$= 4004$ ✓

c) $u = \ln x$ $\quad x = e,\ u = 1$

$du = \frac{1}{x}\,dx$ $\quad x = 1,\ u = 0$ ✓

$\int_1^e \frac{dx}{x(1+2\ln x)^2} = \int_0^1 \frac{du}{(1+2u)^2}$ ✓

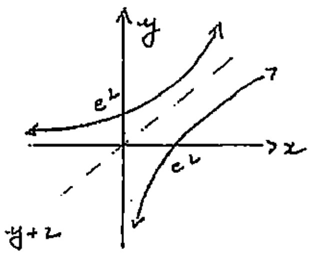
$= \left[\frac{-1}{2}(1+2u)^{-1}\right]_0^1$ ✓

$= \left[\frac{-1}{2(1+2u)}\right]_0^1$

$= \frac{-1}{6} + \frac{1}{2}$

$= \frac{1}{3}$ ✓

6) a) (i)



(iii)

(ii) $x = e^{y+2}$

$y + 2 = \ln x$

$y = \ln x - 2$

b) $v^2 = 32 + 8x - 4x^2$

(i) $\frac{1}{2}v^2 = 16 + 4x - 2x^2$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \ddot{x} = 4 - 4x$

$= -4(x-1)$

which is of the form of S.H.M. $\ddot{x} = -n^2(x - x_1)$

(ii) centre is $x = 1$

(iii) period $T = \pi$ seconds

if $v = 0$, $4(x^2 - 2x - 8) = 0$

$(x-4)(x+2) = 0$

$x = 4, -2$ when $v = 0$

$\therefore$ amplitude $= 3$m

(iv) testing $x = 1 + 3\sin 2t$

when $t = 0$, $x = 1$

testing $x = 1 + 3\cos 2t$

when $t = 0$, $x = 4$

$\therefore x = 1 + 3\cos 2t$ could describe the motion.

(v) $0 = 1 + 3\cos 2t$

$3\cos 2t = -1$

$\cos 2t = -\frac{1}{3}$

Acute angle $= 1.231$ (4 s.f.)

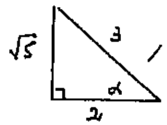
2nd 3rd quadrants

$\therefore\ 2t = \pi - 1.231$

$= 1.9106\ldots$

$\therefore$ first time 0.955 s (approx)

7) a) Let $\cos^{-1}\frac{2}{3} = \alpha$



$\therefore \sin 2\alpha = 2\sin\alpha\cos\alpha$

$= 2\times\frac{\sqrt{5}}{3}\times\frac{2}{3}$

$= \frac{4\sqrt{5}}{9}$ ✓

b) (i) $\sqrt{3}\sin x - \cos x$

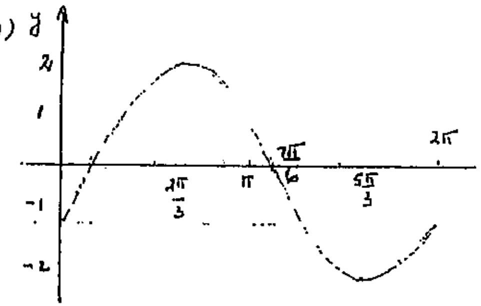
$= A\sin x\cos\alpha - A\cos x\sin\alpha$

$A = 2$, $\therefore \cos\alpha = \frac{\sqrt{3}}{2}$

$\alpha = \frac{\pi}{6}$

$\therefore \sqrt{3}\sin x - \cos x = 2\sin\left(x - \frac{\pi}{6}\right)$

(ii)



(iii) 3 solutions if $k = -1$

c) In ΔPQR,

$\frac{PR}{\sin\angle PQR} = \frac{QR}{\sin\angle QPR}$ (Sine Rule)

$\angle QPR = \angle QSR$ (angles in same segment are equal)

$\therefore \frac{PR}{\sin\angle PQR} = \frac{QR}{\sin\angle QSR}$

In ΔQRS,

$\frac{QR}{\sin\angle QSR} = \frac{QS}{\sin\angle QRS}$ (Sine Rule)

$\therefore \frac{PR}{\sin\angle PQR} = \frac{QS}{\sin\angle QRS}$

$\therefore \frac{PR}{QS} = \frac{\sin\angle PQR}{\sin\angle QRS}$

but $\angle QRS = 180° - \angle QPS$

(opp angles of a cyclic quadrilateral are supplementary)

$\therefore \sin\angle QRS = \sin\angle QPS$

i.e. $\frac{PR}{QS} = \frac{\sin\angle PQR}{\sin\angle QPS}$